On endomorphism algebras of functors with non-compact domain

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Abstract

As a development of [2] and [3], we construct a "VN-core" in \mathbf{Vect}_k for each k-linear split-semigroupal functor from a suitable monoidal category \mathcal{C} to \mathbf{Vect}_k . The main aim here is to avoid the customary compactness assumption on the set of generators of the domain category \mathcal{C} (cf. [3]).

1 Introduction

We propose the construction of a VN-core associated to each (k-linear) split semigroupal functor U from a suitable monoidal category \mathcal{C} to \mathbf{Vect}_k , where all our categories, functors, and natural transformations are assumed to be k-linear, for a fixed field k. Essentially, the category \mathcal{C} must be equipped with a small "U-generator" \mathcal{A} carrying some extra duality information and with UA still being finite dimensional for all A in \mathcal{A} .

We shall use the term "VN-core" (in \mathbf{Vect}_k) to mean a (usual) k-semibialgebra E together with a k-linear endomorphism S such that

$$\mu(\mu \otimes 1)(1 \otimes S \otimes 1)(1 \otimes \delta)\delta = 1 : E \to E.$$

The VN-core is called "antipodal" if S(xy) = SySx (and S(1) = 1) for all $x, y \in E$. This minimal type of structure is introduced here in order to avoid compactness assumptions on the generator $\mathcal{A} \subset \mathcal{C}$ and, at the same time, retain the "fusion" operator, namely

$$(\mu \otimes 1)(1 \otimes \delta) : E \otimes E \to E \otimes E,$$

satisfying the usual fusion equation [7]. Note that here the fusion operator always has a partial inverse (see [1]).

In $\S 2$ we establish sufficient conditions on a functor U in order that

$$\mathrm{End}^{\vee}U = \int^{A} (UA)^* \otimes UA$$

be a VN-core in \mathbf{Vect}_k (following [2]). This core can be completed to a VN-core $\mathrm{End}^{\vee}U \oplus k$ with a unit element. In §3 we give several examples of suitable functors U for the theory.

2 The construction of $\operatorname{End}^{\vee} U$

Let $\mathcal{C} = (\mathcal{C}, \otimes, I)$ be a monoidal category and let

$$U:\mathcal{C} \to \mathbf{Vect}$$

be a functor with both a semigroupal structure, denoted

$$r = r_{C,D} : UC \otimes UD \to U(C \otimes D),$$

and a cosemigroupal structure, denoted

$$i = i_{C,D} : U(C \otimes D) \to UC \otimes UD$$
,

such that ri = 1.

We shall suppose also that there exists a small full subcategory $\mathcal A$ of $\mathcal C$ with the properties:

- 1. UA is finite dimensional for all $A \in \mathcal{A}$,
- 2. *U*-density; the canonical map

$$\alpha_C: \int^A \mathcal{C}(A,C) \otimes UA \to UC$$

is an isomorphism for all $C \in \mathcal{C}$,

3. there is an "antipode" functor

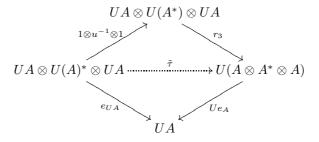
$$(-)^*:\mathcal{A}^{\mathrm{op}} \to \mathcal{A}$$

with a ("canonical") map $e_A: A \otimes A^* \otimes A \to A$ in \mathcal{C} for each $A \in \mathcal{A}$,

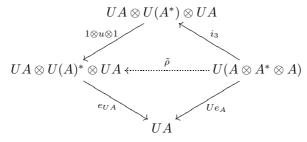
4. there is a natural isomorphism

$$u = u_A : U(A^*) \xrightarrow{\cong} U(A)^*,$$

5. the following diagrams defining $\tilde{\tau}$, $\tilde{\rho}$ both commute



and



where $e_{UA} = 1 \otimes \text{ev}$ in **Vect**, and $r_3 i_3 = 1$.

We now define the semibialgebra structure (End $^{\vee}U, \mu, \delta$) on

$$\operatorname{End}^{\vee} U = \int^{A} U(A)^* \otimes UA$$

as in [2] $\S 2$, with the isomorphism of k-linear spaces

$$S = \sigma : \operatorname{End}^{\vee} U \to \operatorname{End}^{\vee} U$$

given (as in [2] §3) by the usual components

$$U(A)^* \otimes UA \xrightarrow{\sigma_A} U(A^*)^* \otimes U(A^*)$$

$$1 \otimes d \downarrow \qquad \uparrow c$$

$$U(A)^* \otimes U(A)^{**} \xrightarrow{u^{-1} \otimes u^*} U(A^*) \otimes U(A^*)^*$$

where d is the canonical map from a vector space to its double dual. Furthermore, each map

$$e_{UA} = 1 \otimes \text{ev} : UA \otimes UA^* \otimes UA \to UA$$

satisfies both the conditions

$$UA \otimes UA^* \otimes UA$$

$$\downarrow^{n\otimes 1} \qquad \downarrow^{e_{UA}}$$

$$UA \xrightarrow{1} UA$$
(E1)

commutes, and

$$UA^* \otimes UA \otimes UA^*$$

$$1 \otimes n \longrightarrow 1 \otimes d \otimes 1$$

$$UA^* \xrightarrow{e_{UA}^*} UA^* \otimes UA^{**} \otimes UA^*$$
(E2)

commutes, where $n = \text{coev}: 1 \to UA \otimes UA^*$ in **Vect**.

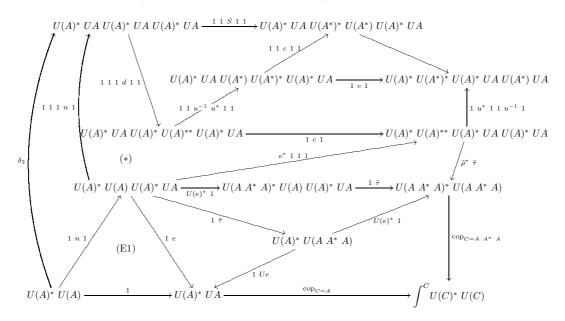
Then we obtain:

Theorem 2.1. The structure $(\operatorname{End}^{\vee} U, \mu, \delta, S)$ is a VN-core in Vect_k which can be completed to the VN-core $(\operatorname{End}^{\vee} U) \oplus k$.

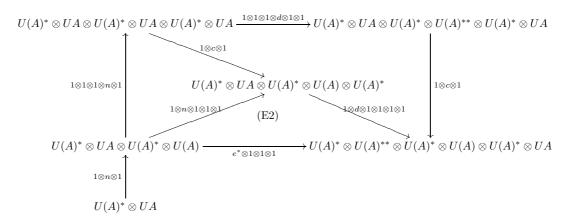
Proof. The von Neumann axiom

$$\mu_3(1\otimes S\otimes 1)\delta_3=1$$

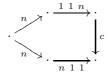
becomes the diagram (in which we have omitted " \otimes "):



where (*) is the exterior of the diagram



which commutes using (E2) and commutativity of



3 Examples

3.1 Example

The first type of example is derived from the idea of a (contravariant) involution on a (small) comonoidal category \mathcal{D} . This includes the doubles $\mathcal{D} = \mathcal{B}^{op} + \mathcal{B}$ and $\mathcal{D} = \mathcal{B}^{op} \otimes \mathcal{B}$ with their respective "switch" maps (where \mathcal{B} is a given small comonoidal \mathbf{Vect}_k -category), or any small comonoidal and compact-monoidal \mathbf{Vect}_k -category \mathcal{D} (such as the category \mathbf{Mat}_k of finite matrices over k) with the tensor duals of objects now providing an antipode on the comonoidal aspect of the structure rather than on the monoidal part, or any *-algebra structure on a given k-bialgebra (e.g., a C^* -bialgebra) with the *-operation providing the antipode.

In each case, an *even* functor from \mathcal{D} to **Vect** is defined to be a (k-linear) functor F equipped with a (chosen) dinatural isomorphism

$$F(D^*) \cong F(D).$$

If we take the morphisms of even functors to be all the natural transformations between them then we obtain a category

$$\mathcal{E} = \mathcal{E}(\mathcal{D}, \mathbf{Vect}).$$

Let $\mathcal{A} = \mathcal{E}(\mathcal{D}, \mathbf{Vect}_{\mathrm{fd}})_{\mathrm{fs}}$ be the full subcategory of \mathcal{E} consisting of the finitely valued functors of finite support. While this category is generally not compact, it has on it a natural antipode derived from those on \mathcal{D} and $\mathbf{Vect}_{\mathrm{fd}}$, namely

$$A^*(D) := A(D^*)^*.$$

Of course, there are also examples where \mathcal{A} is actually compact, such as those where \mathcal{D} is a Hopf algebroid, in the sense of [4], with antipode $(-)^* = S$, in which case each A from \mathcal{D} to **Vect** has a symmetry structure on it.

Now let $\mathcal C$ be the full subcategory of $\mathcal E$ consisting of the small coproducts in $\mathcal E$ of objects from $\mathcal A$. This category $\mathcal C$ is easily seen to be monoidal under the pointwise convolution structure from $\mathcal D$, and the inclusion $\mathcal A \subset \mathcal C$ is U-dense for the functor

$$U: \mathcal{C} \to \mathbf{Vect}_k$$

given by

$$U(C) = \sum_{D} C(D)$$

which is split semigroupal with UA finite dimensional for all $A \in \mathcal{A}$. Moreover,

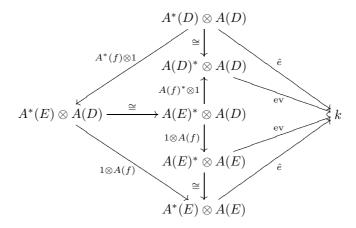
$$U(A^*) = \bigoplus_{D} A^*(D)$$
$$= \bigoplus_{D} A(D)^*$$
$$= U(A)^*.$$

for all $A \in \mathcal{A}$. The conditions of (5) are easily verified if we define maps

$$e:A\otimes A^*\otimes A\to A$$

by commutativity of the diagrams

where the exterior of



commutes for all maps $f:D\to E$ in $\mathcal D$ so that

$$e = 1 \otimes \hat{e} : A \otimes A^* \otimes A \to A \otimes k \cong A$$

is a genuine map in \mathcal{C} when \mathcal{C} is given the pointwise monoidal structure from \mathcal{D} . This completes the details of the general example.

3.2 Example

In the case where $k = \mathbb{C}$ and \mathcal{D} has just one object D whose endomorphism algebra is a C^* -bialgebra, we have a one-object comonoidal category \mathcal{D} with a \mathbb{C} -conjugate-linear antipode given by the *-operation. Then the convolution $[\mathcal{D}, \mathbf{Hilb}_{\mathrm{fd}}]$, where

$$[\mathcal{D},\mathbf{Hilb}_{\mathrm{fd}}]\subset [\mathcal{D},\mathbf{Vect}_{\mathbb{C}}],$$

is a monoidal category, with a \mathbb{C} -linear antipode given by

$$F^*(D) = F(D^*)^{\circ}$$

where H° denotes the conjugate-transpose of $H \in \mathbf{Hilb_{fd}}$. We now interpret an even functor F to be a functor equipped with a dinatural isomorphism $F(D^*) \cong F(D)$ in $D \in \mathcal{D}$ which is \mathbb{C} -linear, so that $F^*(D) \cong F(D)^{\circ}$ for such a functor.

Take $\mathcal{A} = \mathcal{E}(\mathcal{D}, \mathbf{Hilb}_{\mathrm{fd}})$ and let \mathcal{C} be the class of small coproducts in $[\mathcal{D}, \mathbf{Vect}_{\mathbb{C}}]$ of the underlying $[\mathcal{D}, \mathbf{Vect}_{\mathbb{C}}]$ -representations of A's in \mathcal{A} (with the appropriate maps). Each map

$$e: A \otimes A^* \otimes A \to A$$

in \mathcal{C} is defined by the \mathbb{C} -linear components

$$e: A(D) \otimes A^*(D) \otimes A(D) \xrightarrow{1 \otimes \hat{e}} A(D),$$

where

$$\hat{e}: A^*(D) \otimes A(D) \to \mathbb{C}$$

in $\mathbf{Vect}_{\mathbb{C}}$ comes from the \mathbb{C} -bilinear composite of two maps which are both \mathbb{C} -linear in the first variable and \mathbb{C} -linear in the second, namely

$$A^*(D) \times A(D) \xrightarrow{\cong} \mathbb{C}$$

$$\cong \downarrow \qquad (-,-)$$

$$A(D)^{\circ} \times A(D).$$

The remainder of this example is as seen before in Example 3.1.

3.3 Example

Let $\mathcal{V} = (\mathcal{V}, \otimes, I)$ be a (small) braided monoidal category and let \mathcal{B} be the k-linearization of **Semicoalg**(\mathcal{V}) with the monoidal structure induced from that on \mathcal{V} . By analogy with [5], let $\mathcal{X} \subset \mathcal{B}$ be a finite full subcategory of \mathcal{B} with $I \in \mathcal{X}$ and $\mathcal{X}^{\mathrm{op}}$ promonoidal when

$$p(x, y, z) = \mathcal{B}(z, x \otimes y)$$
$$j(z) = \mathcal{B}(z, I)$$

for $x, y, z \in \mathcal{X}$.

For example (cf. [5]), one could take \mathcal{X} to be a (finite) set of non-isomorphic "basic" objects in some braided monoidal category \mathcal{V} , where each $x \in \mathcal{X}$ has a coassociative diagonal map $\delta: x \to x \otimes x$. However, we won't need the category \mathcal{X} to be discrete or locally finite in the following.

Now let \mathcal{C} be the convolution $[\mathcal{X}^{op}, \mathbf{Vect}]$ and let $\mathcal{A} = [\mathcal{X}^{op}, \mathbf{Vect}_{fd}]$. The functor

$$U:\mathcal{C} o \mathbf{Vect}$$

is defined by

$$U(C) = \bigoplus_{x} C(x),$$

and the obvious inclusion $\mathcal{A}\subset\mathcal{C}$ is U-dense. If there is a canonical (natural) retraction

$$p(x,y,z) = \mathcal{B}(z,x \otimes y) \xleftarrow{r_{x,y}} \mathcal{B}(z,x) \otimes \mathcal{B}(z,y),$$

derived from the semicoalgebra structures on x, y, z, then U becomes a split semigroupal functor via the structure maps

$$U(C) \otimes U(D) \xleftarrow{r} U(C \otimes D)$$

$$\parallel \qquad \qquad \qquad \parallel$$

$$\bigoplus_{x} C(x) \otimes \bigoplus_{y} D(y) \qquad \qquad \bigoplus_{z} \int^{xy} p(x,y,z) \otimes C(x) \otimes D(y)$$

$$\triangleq \bigoplus_{z} C(z) \otimes D(z) \xleftarrow{\cong} \bigoplus_{z} \int^{xy} \mathcal{B}(z,x) \otimes \mathcal{B}(z,y) \otimes C(x) \otimes D(y),$$

where the isomorphism follows from the Yoneda lemma, and ri = 1. If \mathcal{X} also has on it a duality

$$(-)^*: \mathcal{X} \to \mathcal{X}^{\mathrm{op}}$$

such that $x \cong x^{**}$, then, on defining

$$A^*(x) = A(x^*)^*,$$

we obtain

$$U(A^*) = \bigoplus_x A^*(x)$$

$$= \bigoplus_x A(x^*)^*$$

$$\cong \bigoplus_x A(x)^*$$

$$\cong U(A)^*,$$
since $x \cong x^{**}$

for $A \in \mathcal{A}$, in accordance with the fourth requirement on U. Finally, to obtain a suitable map

$$e = 1 \otimes \hat{e} : A \otimes A^* \otimes A \to A \otimes I \cong A$$

where $\hat{e}: A^* \otimes A \to I$, we suppose each A in A has on it a "dual coupling"

$$\chi = \chi_{xy} : A(x)^* \otimes A(y) \to \mathcal{B}(x^* \otimes y, I).$$

By considering the Yoneda expansion

$$A(x) \cong \int^z A(z) \otimes \mathcal{X}(x,z)$$

of the various functors A in $\mathcal{A} = [\mathcal{X}^{op}, \mathbf{Vect}_{fd}]$, such a coupling exists on each A if we suppose merely that \mathcal{X} itself is "coupled" by a natural transformation

$$\chi: \mathcal{X}(y,z) \to \mathcal{X}(x,z) \otimes \mathcal{B}(x^* \otimes y,I);$$

or simply

$$\chi: \mathcal{X}(x,z)^* \otimes \mathcal{X}(y,z) \to \mathcal{B}(x^* \otimes y,I),$$

if \mathcal{X} is locally finite. Then, the composite natural transformation

$$A(x^*)^* \otimes A(y) \otimes \mathcal{B}(z, x \otimes y)$$

$$\downarrow^{\chi \otimes 1}$$

$$\mathcal{B}(x^{**} \otimes y, I) \otimes \mathcal{B}(z, x \otimes y)$$

$$\downarrow^{\cong}$$

$$\mathcal{B}(x \otimes y, I) \otimes \mathcal{B}(z, x \otimes y)$$

$$\downarrow^{\text{comp'n}}$$

$$\mathcal{B}(z, I)$$

yields the map

$$A^* \otimes A \xrightarrow{\hat{e}} I$$

$$\parallel$$

$$\int^{xy} A^*(x) \otimes A(y) \otimes p(x, y, -) \longrightarrow \mathcal{B}(-, I)$$

because $p(x, y, -) = \mathcal{B}(-, x \otimes y)$ (by definition). Thus suitable conditions on the coupling χ give (5).

Remark. Actually, this last example in which the basic promonoidal structure occurs as a canonical retract of a comonoidal structure is typical of many other examples which can be treated along similar lines.

References

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